(Conditional) Syntax Splitting, Lexicographic Entailment and the Drowning Effect

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Structure

Motivation

Birds and Mammals

Let $\Delta = \Delta_{\text{birds}} \cup \Delta_{\text{geography}}$ with:

$$\begin{aligned} \Delta_{\text{birds}}: &\quad (\text{birds}|\text{penguins}), (\text{fly}|\text{birds}), (\neg\text{fly}|\text{penguins}) \\ \Delta_{\text{geography}}: &\quad (\text{polar}|\text{antarctic}), (\text{africa}|\text{westernCape}) \end{aligned}$$

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$$\text{penguins} \mid\!\sim_{\Delta} \neg\text{fly}$$

$$\text{penguins} \wedge \text{westernCape} \mid\!\sim_{\Delta} \neg\text{fly}$$

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Birds and Beaks

Let $\Delta = \Delta_{\mathsf{birds}} \cup \Delta_{\mathsf{birds}'}$ with:

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Preliminaries

Background on Propositional Logic and Conditionals

Propositional Logic
$\mathcal{L}(\Sigma)$ constructed on the basis of Σ and $\wedge$, $\vee$, $\neg$ and $\rightarrow$.

Possible worlds $\omega \in \Omega(\Sigma)$ are often denoted as complete conjunctions. E.g. $\overline{p}bf$.

Mod(ϕ) consists of the models of ϕ.

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Conditionals
$(\mathcal{L}|\mathcal{L}) = \{(B|A) \mid A, B \in \mathcal{L}\}$.

$$((B|A))(\omega) = \begin{cases} 1 & \omega \models A \wedge B \\ 0 & \omega \models A \wedge \neg B \\ u & \omega \models \neg A \end{cases}$$

Definition ([KIBB20])
An inductive inference operator (from conditional belief bases) is a mapping $\mathbf{C} : 2^{(\mathcal{L}|\mathcal{L})} \mapsto 2^{\mathcal{L}^2}$ (or, more readable: $\Delta \to {\mid\!\sim}_{\Delta}$) that satisfies:

DI $(B|A) \in \Delta$ implies $A \mid\!\sim_{\Delta} B$.

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Example ($\Delta = \{(f|b)\}$)
$b {\mid\!\sim}_{\Delta} f$,
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$\ldots$

Two examples of inductive inference operators are system Z and lexicographic inference.

Total Preorders [KLM90]

Given a total preorder (in short, TPO) $\preceq$ on possible worlds:

$A \preceq B$ iff $\omega \preceq \omega'$ for an $\omega \in \min_{\preceq}(\mathrm{Mod}(A))$ and an $\omega' \in \min_{\preceq}(\mathrm{Mod}(B))$.

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$$A \mathrel{|\!\sim}_{\preceq} B \text{ iff } (A \wedge B) \prec (A \wedge \neg B).$$

Example

$$\overline{p}bf, \quad \overline{p}\overline{b}f, \quad \overline{p}\overline{b}\,\overline{f} \quad \prec \quad pb\overline{f}, \quad \overline{p}b\overline{f} \quad \prec \dots$$

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Z-ranking of conditionals [GP96]

A conditional $(B|A)$ is tolerated by a finite set of conditionals Δ if there is a possible world ω with:

1. $(B|A)(\omega) = 1$, and
2. $(D|C)(\omega) \neq 0$ for every $(D|C) \in \Delta$.

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The Z-partitioning $(\Delta_0, \ldots, \Delta_n)$ of Δ is defined as:

- $\Delta_0 = \{\delta \in \Delta \mid \Delta \text{ tolerates } \delta\}$;
- $\Delta_1, \ldots, \Delta_n$ is the Z-partitioning of $\Delta \setminus \Delta_0$.

$Z_\Delta(\delta) = i$ iff $\delta \in \Delta_i$.

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$\Delta_0 = \{(f|b)\}$ (in view of $\overline{p}bf$), and $\Delta_1 = \{(b|p), (\neg f|p)\}$.

System Z [GP96]

- $\kappa^Z_\Delta(\omega) = \max\{Z(\delta) \mid \delta(\omega) = 0, \delta \in \Delta\} + 1$, with $\max \emptyset = -1$.
- $A \mid\!\sim^Z_\Delta B$ iff $A \mid\!\sim_{\kappa^Z_\Delta} B$.

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Recall: $\Delta_0 = \{(f|b)\}$ and $\Delta_1 = \{(b|p), (\neg f|p)\}$.

ω	κ^Z_Δ	ω	κ^Z_Δ	ω	κ^Z_Δ	ω	κ^Z_Δ
pbf	2	$pb\overline{f}$	1	$p\overline{b}f$	2	$p\overline{b}\,\overline{f}$	2
$\overline{p}bf$	0	$\overline{p}b\overline{f}$	1	$\overline{p}\overline{b}f$	0	$\overline{p}\overline{b}\,\overline{f}$	0

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$$\top \mid\!\sim^Z_\Delta \neg p.$$
$$p \wedge f \not\mid\!\sim^Z_\Delta b.$$

Lexicographic Inference [Leh95]

- Given $\omega \in \Omega$ and $\Delta' \subseteq \Delta$,
  $V(\omega, \Delta') = |(\{(B|A) \in \Delta' \mid (B|A)(\omega) = 0\}|$.

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- Given $\omega \in \Omega$ and $\Delta' \subseteq \Delta$,
  $V(\omega, \Delta') = |(\{(B|A) \in \Delta' \mid (B|A)(\omega) = 0\}|$.
- The lexicographic vector for ω is:
  $\mathrm{lex}(\omega) = (V(\omega, \Delta_0), \ldots, V(\omega, \Delta_n))$.
- Given two vectors $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_n)$,
  $(x_1, \ldots, x_n) \preceq^{\mathrm{lex}} (y_1, \ldots, y_n)$ iff there is some $j \leq n$ s.t.
  $x_k = y_k$ for every $k > j$ and $x_j \leq y_j$.
- $\omega \preceq^{\mathrm{lex}}_{\Delta} \omega'$ iff $\mathrm{lex}(\omega) \preceq^{\mathrm{lex}} \mathrm{lex}(\omega')$.

Lexicographic Inference [Leh95]

Example ($\Delta = \{(f|b), (b|p), (\neg f|p)\}$)

ω	$\text{lex}(\omega)$	ω	$\text{lex}(\omega)$	ω	$\text{lex}(\omega)$	ω	$\text{lex}(\omega)$
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$\overline{p}bf$	(0,0)	$\overline{p}b\overline{f}$	(1,0)	$\overline{p}\overline{b}f$	(0,0)	$\overline{p}\overline{b}\,\overline{f}$	(0,0)

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$$\top \mid\!\sim^{\text{lex}}_{\Delta} \neg p.$$

$$p \wedge f \mid\!\sim^{\text{lex}}_{\Delta} b.$$

Syntax Splitting

Splitting Conditional Belief Bases [KIBB20]

We assume a conditional belief base Δ that can be split into subbases Δ_1, Δ_2 s.t. $\Delta_i \subset (\mathcal{L}_i|\mathcal{L}_i)$ with $\mathcal{L}_i = \mathcal{L}(\Sigma_i)$ for $i = 1, 2$ s.t. $\Sigma_1 \cap \Sigma_2 = \emptyset$ and $\Sigma_1 \cup \Sigma_2 = \Sigma$, writing:

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Example

$$\{(a|\top), (b|\top)\} = \{(a|\top)\} \bigcup_{\{a\},\{b\}} \{(b|\top)\}$$

Independence [KIBB20]

Definition (Independence (Ind))
An inductive inference operator **C** satisfies (**Ind**) if for any $\Delta = \Delta^1 \bigcup_{\Sigma_1, \Sigma_2} \Delta^2$ and for any $A, B \in \mathcal{L}_i$, $C \in \mathcal{L}_j$ $(i, j \in \{1, 2\}$, $j \neq i)$,

$$A \mid\!\sim_{\Delta} B \text{ iff } AC \mid\!\sim_{\Delta} B$$

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Example ($\Delta = \{(a|\top), (b|\top)\}$)

$\top \mid\!\sim_\Delta a$	**DI**
$b \mid\!\sim_\Delta a$	**Ind**

Relevance

Definition (Relevance (Rel))
An inductive inference operator **C** satisfies (**Rel**) if for any $\Delta = \Delta^1 \bigcup_{\Sigma_1,\Sigma_2} \Delta^2$ and for any $A, B \in \mathcal{L}_i$ $(i \in \{1, 2\})$,

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Definition (Relevance (Rel))
An inductive inference operator **C** satisfies (**Rel**) if for any $\Delta = \Delta^1 \bigcup_{\Sigma_1,\Sigma_2} \Delta^2$ and for any $A, B \in \mathcal{L}_i$ $(i \in \{1,2\})$,

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Example ($\Delta = \{(a|\top),(b|\top)\}$)

$\top \mid\!\sim_{\Delta} a$	**DI**
$\top \mid\!\sim_{\{(a\|\top)\}} a$	**Rel**

Syntax Splitting= Independence + Relevance

Definition (Syntax-Splitting (SynSplit))
An inductive inference operator **C** satisfies (**SynSplit**) if it satisfies (**Ind**) and (**Rel**).

Proposition

C^{lex} and C^{Z} satisfy **Rel**.

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C^{lex} satisfies **Ind**.

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C^{lex} and C^{Z} satisfy **Rel**.

Proposition

C^{lex} satisfies **Ind**.

Proposition
C^{Z} does not satisfy **Ind**.

Example
Let $\Delta = \{(a|\top), (b|\top)\}$. Then:

$$ab \prec^{Z}_{\Delta} a\overline{b}, \overline{a}b, \overline{a}\overline{b} \qquad ab \prec^{\text{lex}}_{\Delta} a\overline{b}, \overline{a}b \prec^{\text{lex}}_{\Delta} \overline{a}\overline{b}$$

$$\top \mid\!\sim^{Z}_{\Delta} a \qquad \neg b \not\mid\!\sim^{Z}_{\Delta} a \qquad \top \mid\!\sim^{\text{lex}}_{\Delta} a \qquad \neg b \mid\!\sim^{\text{lex}}_{\Delta} a$$

The Drowning Effect

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Tweety-knowledge base together with the fact that **b**irds typically have b**e**aks:

$$\{(f|b),(b|p),(\neg f|p),(e|b)\}$$

Do penguins have beaks: $p \mathrel{|\!\sim}_{\Delta} e$?

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According to system Z, not: $\kappa^Z_\Delta(pb\overline{f}e) = \kappa^Z_\Delta(pb\overline{f}\overline{e})$

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According to system Z, not: $\kappa^Z_\Delta(pb\overline{f}e) = \kappa^Z_\Delta(pb\overline{f}\overline{e})$ (since both worlds falsify the rule $(f|b) \in \Delta_0$).

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According to system Z, not: $\kappa^{Z}_{\Delta}(pb\overline{f}e) = \kappa^{Z}_{\Delta}(pb\overline{f}\overline{e})$ (since both worlds falsify the rule $(f|b) \in \Delta_0$).

According to lexicographic inference, they do (since $\mathrm{lex}(pb\overline{f}e) = (0,1) \prec_{\mathrm{lex}} (1,1) = \mathrm{lex}(pb\overline{f}\overline{e})$).

Syntax Splitting and the Drowning Effect

For syntax splitting to be applied, we need full syntactic separation of the syntax of a knowledge base:

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$$\{(f|b),(b|p),(\neg f|p),(e|b)\}$$

In the paper, we define an inductive inference relation that satisfies syntax splitting yet suffers from the drowning effect.

I.e. the drowning effect is independent of syntax splitting.

Conditionally Splitting a Knowledge Base

Conditional Splitting: naive attempt

Definition
We say a conditional belief base Δ can be *split into subbases* Δ_1,Δ_2 *conditional on a sub-alphabet* Σ_3, if $\Delta_i \subset (\mathcal{L}(\Sigma_i \cup \Sigma_3) \mid \mathcal{L}(\Sigma_i \cup \Sigma_3))$ for $i = 1, 2$ s.t. Σ_1, Σ_2 and Σ_3 are pairwise disjoint and $\Sigma = \Sigma_1 \cup \Sigma_2 \cup \Sigma_3$, writing:

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Definition
We say a conditional belief base Δ can be *split into subbases* Δ_1, Δ_2 *conditional on a sub-alphabet* Σ_3, if $\Delta_i \subset (\mathcal{L}(\Sigma_i \cup \Sigma_3) \mid \mathcal{L}(\Sigma_i \cup \Sigma_3))$ for $i = 1, 2$ s.t. Σ_1, Σ_2 and Σ_3 are pairwise disjoint and $\Sigma = \Sigma_1 \cup \Sigma_2 \cup \Sigma_3$, writing:

$$\Delta = \Delta^1 \bigcup_{\Sigma_1, \Sigma_2} \Delta^2 \mid \Sigma_3$$

Example ($\Delta = \{(x|b), (\neg x|a), (c|a \wedge b)\}$)
Then

$$\Delta = \{(x|b), (\neg x|a)\} \bigcup_{\{x\},\{c\}} \{(c|a \wedge b)\} \mid \{a, b\}$$

$(c|a \wedge b)$ (trivially) tolerates itself, yet Δ does not tolerate $(c|a \wedge b)$, i.e. $Z_\Delta((c|a \wedge b)) = 1 \neq Z_{\Delta^2}((c|a \wedge b))$.

Safe Conditional Splitting

Definition

A conditional belief base $\Delta = \Delta^1 \bigcup_{\Sigma_1,\Sigma_2} \Delta^2 \mid \Sigma_3$ can be *safely split into subbases* Δ_1,Δ_2 *conditional on a sub-alphabet* Σ_3, writing:

$$\Delta = \Delta^1 \overset{s}{\bigcup_{\Sigma_1,\Sigma_2}} \Delta^2 \mid \Sigma_3$$

if for every $\omega^3 \in \Omega(\Sigma_3)$, there is a $\omega^j \in \Omega(\Sigma_j)$ s.t. $\omega^j\omega^3 \models \bigwedge_{(F|E)\in\Delta^j} E \rightarrow F$ (for $i, j = 1, 2$ and $i \neq j$).

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Intuition: any information about $\Sigma_i \cup \Sigma_3$ is compatible with Δ^j. In other words, toleration with respect to Δ^j is independent of Δ^i.

Safe Conditional Splitting: Example

$$\Delta = \{(f|b), (b|p), (\neg f|p)\} \bigcup^{s}_{\{p,f\},\{e\}} \{(e|b)\} \mid \{b\}.$$

since:

- for $\omega^3 = b$, $b\overline{p}f \models (b \to f) \wedge (p \to b) \wedge (p \to \neg f)$
- for $\omega^3 = \overline{b}$, $\overline{b}\overline{p}f \models (b \to f) \wedge (p \to b) \wedge (p \to \neg f)$
- and similarly for $\{(e|b)\}$.

Safe Conditional Splitting and Toleration

Proposition

Let a conditional belief base $\Delta^1 \bigcup^{s}_{\Sigma_1,\Sigma_2} \Delta^2 \mid \Sigma_3$ be given. Then for any $i = 1, 2$:

Δ^i *tolerates* $(B|A) \in \Delta^i$ *iff* Δ *tolerates* $(B|A)$.

Conditional Syntax Splitting

Definition
An inductive inference operator **C** satisfies (**CInd**) if for any $\Delta^1 \bigcup^{s}_{\Sigma_1,\Sigma_2} \Delta^2 \mid \Sigma_3$, and for any $A, B \in \mathcal{L}(\Sigma_i)$, $C \in \mathcal{L}(\Sigma_j)$ (for $i, j \in \{1, 2\}$, $j \neq i$) and a complete conjunction $D \in \mathcal{L}(\Sigma_3)$,

$$AD \mathrel{|\!\sim}_{\Delta} B \text{ iff } ADC \mathrel{|\!\sim}_{\Delta} B$$

Conditional Independence: Example

$$\Delta = \{(f|b), (b|p), (\neg f|p)\} \bigcup_{\{p,f\},\{e\}} \{(e|b)\} \mid \{b\}.$$

$$p \wedge b \mid\!\sim_{\Delta} \neg f \quad \text{iff} \quad p \wedge e \wedge b \mid\!\sim_{\Delta} \neg f$$

Conditional Independence: Example

$$\Delta = \{(f|b), (b|p), (\neg f|p)\} \bigcup_{\{p,f\},\{e\}} \{(e|b)\} \mid \{b\}.$$

$$p \wedge b \mid\!\sim_{\Delta} \neg f \quad \text{iff} \quad p \wedge e \wedge b \mid\!\sim_{\Delta} \neg f$$

$$b \mid\!\sim_{\Delta} e \quad \text{iff} \quad p \wedge b \mid\!\sim_{\Delta} e$$

Conditional Independence for TPOs

Proposition

An inductive inference operator for TPOs $\mathbf{C}^{tpo} : \Delta \mapsto \preceq_\Delta$ on $\mathcal{L}$ satisfies (**CInd**) iff for any $\Delta = \Delta^1 \bigcup^{s}_{\Sigma_1,\Sigma_2} \Delta^2 \mid \Sigma_3$, it holds for all $i, j \in \{1, 2\}, i \neq j$, that:

$$\omega_1^i \omega_1^j \omega^3 \preceq \omega_2^i \omega_1^j \omega^3 \text{ iff } \omega_1^i \omega^3 \preceq \omega_2^i \omega^3. \quad (1)$$

Conditional Relevance

Definition
An inductive inference operator **C** satisfies (**CRel**) if for any $\Delta = \Delta^1 \bigcup^{s}_{\Sigma_1,\Sigma_2} \Delta^2 \mid \Sigma_3$, and for any $A, B \in \mathcal{L}(\Sigma_i)$, $C \in \mathcal{L}(\Sigma_j)$ (for $i, j \in \{1, 2\}$, $j \neq i$) and a complete conjunction $D \in \mathcal{L}(\Sigma_3)$,

$$AD \mid\!\sim_{\Delta} B \text{ iff } AD \mid\!\sim_{\Delta_i} B$$

Conditional Relevance: Example

$$\Delta = \{(f|b), (b|p), (\neg f|p)\} \bigcup_{\{p,f\},\{e\}} \{(e|b)\} \mid \{b\}.$$

$$p \wedge b \mathrel{|\!\sim}_{\Delta} \neg f \quad \text{iff} \quad p \wedge b \mathrel{|\!\sim}_{\Delta^1} \neg f$$

Conditional Relevance: Example

$$\Delta = \{(f|b), (b|p), (\neg f|p)\} \bigcup_{\{p,f\},\{e\}} \{(e|b)\} \mid \{b\}.$$

$$p \wedge b \mathrel{|\!\sim}_{\Delta} \neg f \quad \text{iff} \quad p \wedge b \mathrel{|\!\sim}_{\Delta^1} \neg f$$

Lexicographic Inference Satisfies Conditional Syntax Splitting

Proposition

C^{lex} satisfies **CInd** and **CRel**.

Proposition

C^{lex} satisfies **CInd** and **CRel**.

The crucial result is this:

Lemma

Let a conditional belief base $\Delta^1 \bigcup^{s}_{\Sigma_1,\Sigma_2} \Delta^2 \mid \Sigma_3$ with its corresponding Z-partition $(\Delta_0, \ldots, \Delta_n)$ be given. Then for every $0 \leq i \leq n$:

$$\begin{aligned} V(\omega, \Delta_i) &= V(\omega^1\omega^3, \Delta^1_i) + V(\omega^2\omega^3, \Delta^2_i) - V(\omega^3, \Delta^1_i) \\ &= V(\omega^1\omega^3, \Delta^1_i) + V(\omega^2\omega^3, \Delta^2_i) - V(\omega^3, \Delta^2_i) \end{aligned}$$

Conditional Independence and the Drowning Effect

$$\Delta = \{(f|b), (b|p), (\neg f|p)\} \bigcup_{\{p,f\},\{e\}} \{(e|b)\} \mid \{b\}.$$

$$b \mathrel{|\!\sim}_{\Delta} e \qquad \text{by } \mathbf{DI} \tag{2}$$

$$b \wedge p \mathrel{|\!\sim}_{\Delta} e \qquad \text{by } \mathbf{CInd} \text{ and (2)} \tag{3}$$

For any inductive inference operator that additionally satisfies **Cut** we obtain:

$$p \mathrel{|\!\sim}_{\Delta} b \qquad \text{by } \mathbf{DI} \tag{4}$$

$$p \mathrel{|\!\sim}_{\Delta} e \qquad \text{by } \mathbf{Cut}\text{, (3) and (4)} \tag{5}$$

Conditional Independence and the Drowning Effect: more general

- Do exceptional subclasses (e.g. penguins) inherit properties of a superclass (e.g. birds), even if these properties are unrelated to the reason for the subclass being exceptional (e.g. having beaks)?
- *Unrelatedness of propositions* can formally captured by *safe splitting into subbases*:

  Given a belief base Δ, a proposition A is unrelated to a proposition C iff Δ can be safely split into subbases Δ_1, Δ_2 conditional on a sub-alphabet Σ_3, i.e. $\Delta = \Delta^1 \bigcup^{s}_{\Sigma_1,\Sigma_2} \Delta^2 \mid \Sigma_3$, and $A \in \mathcal{L}(\Sigma_2)$ and $C \in \mathcal{L}(\Sigma_1 \cup \Sigma_3)$.

- The drowning effect is nothing else than a violation of (**CInd**):

  if a typical property B of AD-individuals ($AD \mid\!\sim_{\Delta} B$) is unrelated to an exceptional subclass C of AD, then we can also derive that if something is ADC is typically B

Future Work

- ✓ Lehmann's "desirable properties" [Leh95] are also consequences of conditional independence.
- ? Are there other TPO-based inference operators that satisfy conditional syntax splitting?
- ? Do c-representations and system W satisfy conditional syntax splitting?
- ? Can we give an axiomatic characterization of lexicographic inference?
- ? How to discover conditional independencies?
- ? Implementations.

Conclusion

- Lexicographic inference satisfies syntax splitting as defined in [KIBB20].
- Syntax splitting is independent from the drowning effect.
- Avoidance of the drowning effect is implied by conditional syntax splitting (not previously formulated in the literature).
- Lexicographic inference satisfies conditional syntax splitting.

Conclusion

- Lexicographic inference satisfies syntax splitting as defined in [KIBB20].
- Syntax splitting is independent from the drowning effect.
- Avoidance of the drowning effect is implied by conditional syntax splitting (not previously formulated in the literature).
- Lexicographic inference satisfies conditional syntax splitting.

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